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A Dynamic Houjin (Square) and a Symmetric Houjin

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Abstract: A Houjin is an *n* by *n* square lattice with each cell containing a symbol (such as a number or a letter). Further, these numbers or letters are designed to exhibit symmetry. For example, a magic square is a Houjin where the symmetry embedded is that the numbers in each row, column and a center diagonal have an equal sum. This paper reports a new Houjin: a dynamic Houjin. A dynamic Houjin changes its containing numbers at each time step while satisfying the symmetry as the Houjin (the magic square). The dynamic Houjin has a further symmetry in a time dimension, that is, the sums of the numbers for each cell are identical.

Keywords: Houjin, magic square, Latin square, Euler square, Sudoku**.**

I. INTRODUCTION

Houjin in Japanese mathematics means a magic square (or *Lo Shu* in Chinese). A Houjin is an *n* by *n* square lattice with each cell containing a number where these numbers satisfy a certain constraint: sums in each row, column, and a center diagonal are equal. More general Houjin can have other symmetries.

Several Houjins (squares) have been studied in discrete mathematics, typified by Latin squares and Greco-Latin squares (Euler squares). As several names such as Houjin, Lo Shu, and Squares indicated, they have been extensively studied in world wide ([1-4] in Japan for example) and for a period of historic scale. Greco-Latin squares have been used in the experimental design to make sure all the possible combination of the control factors are involved. As for Latin square, 9x9 two layered latin square are used a puzzle known as Sudoku [5].

On the other hand, cellular automata (CA) have been attracting attention as a potential model for complex systems (e.g., [6]). CA also can be expressed as a square lattice where each cell can take one state among several states. One difference between CA and Houjin is that the former is a dynamical system, while the latter is a static one.

In order to bridge between CA and Houjin, this paper tries a preliminary study to design a dynamic Houjin that changes by a certain rule while satisfying the symmetry of Houjin. Since there is much degree of freedom in a design of the dynamic Houjin, this paper can present an example of the dynamic Houjin based on symmetric Houjins.

Section II briefly states the studies of Houjin focusing on the *spatially* elaborated ones to contrast *temporally* devised ones: the dynamic Houjin. Among spatial Houjins, symmetric Houjins are used to demonstrate the dynamic Houjin. Section III presents the dynamic Houjin, giving an example of the one constructed from 4×4 symmetric Houjin.

II. HOUJIN AS COMPUTATIONAL AND MATHEMATICAL OBJECTS

Houjin have been discussed as a mathematical objects, however, it could be placed as a computational object. For example, Houjin with two numbers in each cell may be related to a matching problem: Stable Marriage Problem (SMP) [7]. An SMP of *n* men and *n* women requires two preference matrices: one indicating each man's preference over the set of n women; and another indicating each woman's preference over the set of n men. These two matrices may be combined as a Houjin with two-tuple of numbers (m_{ij},,w_{ij})where m_{ij} is the preference of man *i* to woman *j* and w_{ij} that of woman *j* to man *i* in each cell. A Houjin M={ m_{ij} } (W={ w_{ij} }) is a half Latin Houjin, since each number appears once and only once in each row (column). When these two Houjins are orthogonal the combined Houjin becomes a Greco-Latin Houjin (Euler Houjin).

1. Recursive Houjin

Recursive Houjin is defined to be a Houjin whose structure is the same one with another row and column added or the one deleted. For example, a *recursive Houjin*, can be generated by the SMP above. When the most popular man and woman, hence they are mutually the first order with each other, are added the resulted

Houjin with two-tuple is the recursive houjin (new row and column is added at the upper left corner).

1/1

1/1	2/1
1/2	2/2

1/1	2/1	3/1
1/2	2/2	3/2
1/3	2/3	3/3

Fig.1. Recursive Houjin

Houjin with some specific structure resulted from SMP can be used to study not only SMP (such as the Latin SMP [7]) but also Houjin itself such as the Euler Square.

2. Symmetric Houjin

The numbers in a Houjin can have symmetry. For example, the numbers of a pair of cells in a rotationally symmetric position can be complement. In 4×4 Houjin, for example, a complement numbers are two numbers that sum up to 17 such as 1 and 16; 12 and 5. Fig. 2 shows three examples of 4×4 symmetric Houjin.

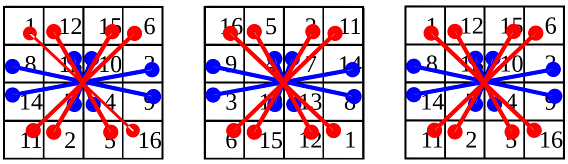

Fig.2. Three examples of 4×4 symmetric Houjin Complement numbers are connected by the edge.

III. DYNAMIC HOUJIN

Dynamic Houjin can be considered as a special type of three dimensional Houjin whose shapes may not necessarily be cubic and one dimension is a specific type: time. An interesting property of the Dynamic Houjin proposed here satisfies the time sum constant is the specific number; and its partition is realized at each cell. Further, in three dimension Houjin of size N, N^3 distinct numbers appear, although the dynamic Houjin proposed here uses N^2 distinct numbers and the the number set do not change as the time proceeds.

Dynamic Houjin may be placed as a special type of cellular automata where the next state of each cell does not necessarily depend on the states of neighbor cells; rather depend on more global configuration of cell states. Further, it must satisfy the global constraints such as the constant sum for several directions (row, column, center diagonal and pan-diagonal).

Dynamic Houjin is a new type of Houjin that changes as time step proceeds satisfying the two constrains: it must satisfy the constraint of the initial Houjin; and it has a constraint along the time dimension similar to the spatial constraint of the initial Houjin For example, the dynamic Houjin discussed here satisfies the constraint satisfied by the magic square: the equal sum in row, column, and center diagonal at each step; and at some specific time step the sum of every cell must be identical. We can build an example of 4×4 dynamic Houjin based on 4×4 symmetric Houjin.

When defining dynamic Houjin, an updating rule for generating the next Houjin from the current Houjin is required. A simple updating rule, for this case is exchanging the number in a symmetric position in the square (Fig. 3). Since the numbers in a symmetric position of the symmetric Houjin form a complement of 17, this updating rule defines the dynamic Houjin of the period two. That is, it turns back the original Houjin in two steps; and further the numbers in each cell adds up to 17. We call two Houjins are *complement* when the numbers in one Houjin are complement to the numbers in the same cell in another Houjin.

1	12	15	6
8	13	10	3
14	7	4	9
11	2	5	16

S(0) at time=0;

16	5	2	11
9	4	7	14
3	10	13	8
6	15	12	1

S(1)= S(0)' at time=1

Fig.3. A dynamic Houjin with period two.

A dynamic Houjin with the period of 16 can be considered when four operators of exchanging rows and columns are involved (Fig. 4). We denote a Houjin by a number *n* and its complement by *n'* and its horizontal (vertical) mirror image *nX* (*nY*) (Fig. 5). A Houjin *nX* is complement to *nY*. When the initial Houjin is S(0) (Fig. 3 left) and the four operators R1, C1, R2 and C2 are applied four times in this order, the trajectory of the dynamic Houjin is shown in Fig. 6. The four operators are depicted by colored arrows.

Note that the Houjin that appears after applying the four operators twice is the Houjin complement to the initial Houjin. That is, the Houjins in a symmetric position in the circle trajectory are complement with each other. Also, a Houjin *nX* (*nY*) can be obtained by applying the four operators R1R2R1R2 (C1C2C1C2) to a Houjin *n*.

Starting from the same initial Houjin but changing the orders of the application of four operators, several trajectories with distinct symmetric Houjins are obtained (Fig. 7). The trajectory *0-1-9-12-15Y-14X-10X-4'-0'-1'-9'-12'-15X-14Y-10Y-4-0* is the same trajectory as that shown in Fig. 6. Fig. 7 shows a system of symmetric Houjins, including all possible trajectory when the four operators R1, R2, C1, C2 are applied in an arbitrary order. It can be observed that this system may be considered a development chart of torus, since a torus (not sphere) will be formed by connecting corresponding Houjins in the boundary.

Another system that includes Houjins *nR* instead of *n* exists where *nR* is the 90 degree clock-wise rotation of the Houjin *n*. In this system, *nR*, *nXR*, *nYR*, and *n'R* appear instead of *n*, *nX*, *nY*, and *n'*, respectively.

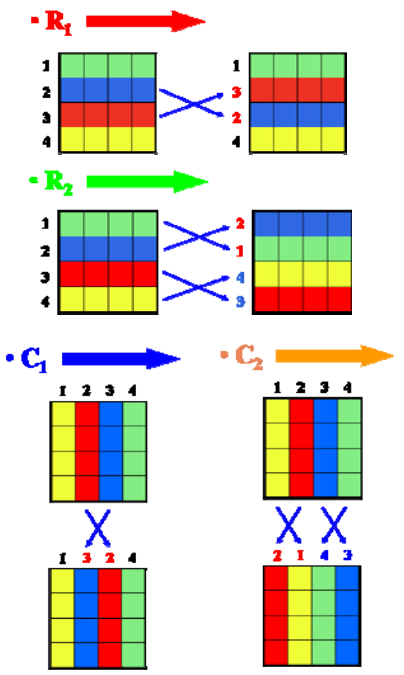

Fig.4. Four operators to change Houjin

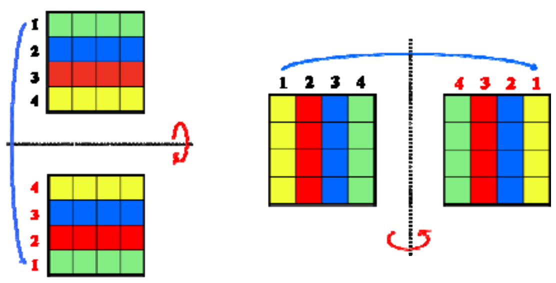

Fig.5. A Houjin *n* and its horizontal (left) a nd vertical (right) mirror image *nX* and *nY*)

VI. DISCUSSIONS

The study of a dynamic Houjin suggested the Houjins themselves would exhibit an external symmetry, although the studies Houjin mainly focused on the internal symmetry embedded in the numbers of a Houjin. For example, the trajectory of the dynamic Houjin (Fig. 6) indicated Houjins in a symmetric position form a complement of Houjin, while in a symmetric Houjin the numbers in a pair of numbers in the cells in a symmetric position form a complement.

Another interesting property of the dynamic Houjin is that the numbers in the same cell add up to 136 which is the sum of all the numbers from 1 to 16 (that is 17x16/2). However, which numbers (among 1 to 16) appears and how many times in a period depend on the cell, suggesting a possible relation to the partition number.

Further, the dynamic Houjin proposed here is a specific type whose trajectory proceeds with the same set of operators applied repeatedly in a fixed order of operations; and the trajectory of a period is divided into two parts in which Houjins appear in a fixed order in the first half and then the complement Houjins appear in the reversed order in the last half. Various and more general trajectory and the way of change (application of operators) could be found.

VI. CONCLUSION

An attempt to develop new types of Houjin (square) has been made with an emphasis that Houjins would have external symmetry in the relation among Houjins other than the internal symmetry so far studied.

Further, since Houjin is a mathematical object similar to numbers, each Houjin could have a specific character similarly to a specific numbers. Composing new type of Houjin is a design mathematics that is required for education in current Engineering and Science.

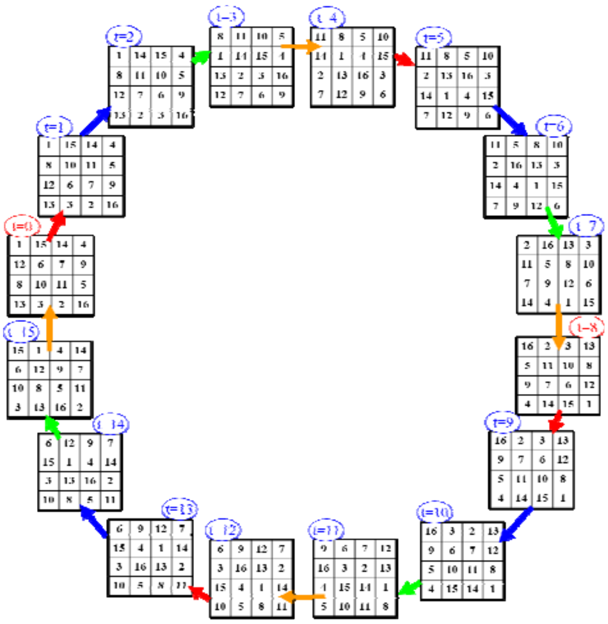

Fig.6. Trajectory of a dynamic Houjin starting fro m S(0) and four operators R1, C1, R2 and C2 a re applied in this order.

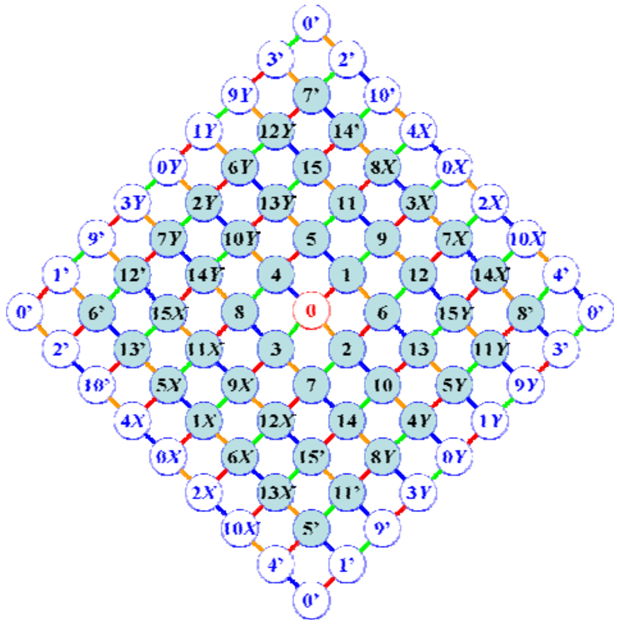

Fig.7. A system of symmetric Houjins, includin g all possible trajectory when the four operator

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